Limit Techniques

Definition of Limit (Somewhat informal)

If $\lim_{x\to a} f(x) = L$, then as x gets closer and closer to a from either the left or right, $f(x)$ gets closer and closer to L.

Things to try

1. *Substitution – Try this first!*

Substitute the limit value into the function and see if you get a sensible answer.

$$\lim_{x\to 5} 2x^2 + 4x - 6 \;\to\; 2(5)^2 + 4(5) - 6 = 64$$

Results with special meanings: $\frac{n}{\infty} = 0$

- $\frac{\pm\infty}{n} = \infty$ *or* $-\infty$
- $\frac{n}{0} = \infty, -\infty,$ *or* DNE
- $\frac{\pm\infty}{\pm\infty}$ or $\frac{0}{0}$ ← do something else (items 2–6, below)

2. *Factor, reduce, then substitute* (Rational Expressions)

$$\lim_{x\to -3} \frac{(x-7)(x+3)}{(x+3)} = \lim_{x\to -3}(x-7) = -10$$

3. *Multiply by the Conjugate, then substitute* (Rational Expressions with radicals)

$$\lim_{x\to 9} \frac{(x-9)}{\sqrt{x}-3} \;\to\; \lim_{x\to 9} \frac{(x-9)}{\sqrt{x}-3}\,\frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9} = \lim_{x\to 9}(\sqrt{x}+3) = \sqrt{9}+3 = 6$$

4. *Simplify complex fractions*

$$\lim_{x\to 2} \frac{\frac{2}{x-1} - \frac{4}{x}}{(x-2)} \;\to\; \lim_{x\to 2} \frac{\frac{2}{x-1} - \frac{4}{x}}{(x-2)}\,\frac{x(x-1)}{x(x-1)} \;\to\; \lim_{x\to 2} \frac{2x-4(x-1)}{x(x-2)(x-1)} \;\to\; \lim_{x\to 2} \frac{2x-4x+4}{x(x-2)(x-1)}$$

5. *Use the trig theorems* (from the sidebar)

The x in the sidebar can be any expression as long as it approaches zero at the limit. Multiply top and bottom by whatever is needed to make the denominator match the trig function's argument.

$$\lim_{x\to 0} \frac{\sin 2x}{3x} \;\to\; \lim_{x\to 0} \frac{2/3}{2/3} \cdot \frac{\sin 2x}{3x} = \lim_{x\to 0} \frac{\frac{2}{3}\sin 2x}{2x} = \tfrac{2}{3} \cdot 1 = \tfrac{2}{3}$$

Three Theorems

These come up a lot:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

$$\lim_{x\to 0} \left(1 + \tfrac{1}{x}\right)^x = e$$

6. Apply Squeeze Theorem, if possible

Find two numbers or two functions, one of which is always above and one of which is always below the function whose limit you are trying to find. If the values of limits of those two functions are equal at your target *x* value, then that value is the limit of your target function.

