Acumen Tutoring™

Calculus Memorize Me™

ε-δ Proof of Limits

for Quadratic Equations

Epsilon-Delta Definition of Limit

$\lim_{x \to a} f(x) = L$ is true if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all x,

if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

In other words, if you want $f(x)$ to be within a particular very small distance (ε) of L, then however small ε may be, there will be a value of x within a small distance (δ) of a, that will make it so.

How to Prove a Limit

Problem: Prove $\lim_{x \to 3} (x^2 - 4) = 5$

Part 1: The preliminary: Determine δ in terms of ε

1 Start with the definition as it applies to our specific function and limit.

$|x^2 - 4 - 5| < \varepsilon$ $\quad |x - 3| < \delta$ $\quad$ (ε and δ are assumed to be positive)

2 Reduce and factor the *epsilon* equation

$$|x^2 - 9| < \varepsilon$$

$$|x - 3||x + 3| < \varepsilon$$

3 Solve for the δ expression.

$$|x - 3| < \frac{\varepsilon}{|x + 3|}$$

4 Assume that $|x - 3| < 1$. From this it follows that

$$-1 < x - 3 < 1$$

$$2 < x < 4$$

$$5 < x + 3 < 7$$

$|x - 3| < \frac{\varepsilon}{7}$ $\quad$ Since $\frac{\varepsilon}{|x + 3|}$ is smallest when $|x + 3|$ is largest.

5 We now have two possible conditions for δ:

$\delta < \frac{\varepsilon}{7}$ $\quad \delta < 1$

which we can summarize as

$$\delta < min(1, \frac{\varepsilon}{7})$$

Part 2: The Proof of $\lim_{x\to 3}(x^2 - 4) = 5$

Because part 1 left us with

$$\delta < min(1, \frac{\varepsilon}{7})$$

we need to test two cases: $\delta = 1$ and $\delta = \frac{\varepsilon}{7}$. We need to demonstrate that *both* of these cases lead to $|x^2 - 9| < \varepsilon$, which for us is $|f(x) - 5| < \varepsilon$ (see step 2 on the previous page).

Case 1: $\delta = 1$

Let $\delta = 1$; this presumes that $1 < \frac{\varepsilon}{7}$, from which we know $7 < \varepsilon$.

Let $|x - 3| < \delta$ 'Cause this is the meaning of δ

$|x - 3| < 1$ Substitution

$|x - 3||x + 3| < |x + 3|$ Multiply by $|x + 3|$ (Remember, we're trying to get to $|x^2 - 9|$)

$|x^2 - 9| < |x + 3|$

Now for some tricky reasoning: remember back in part 4, on the previous page, we said

$$5 < x + 3 < 7, \text{ therefore...}$$

$|x^2 - 9| < 7$ and since $7 < \varepsilon$ (it's right there at the start of our proof)

$|x^2 - 9| < \varepsilon$

$|x^2 - 4 - 5| < \varepsilon$

$|f(x) - 5| < \varepsilon$ This is where we want to be.

Case 2: $\delta = \frac{\varepsilon}{7}$

Let $\delta = \frac{\varepsilon}{7}$

Let $|x - 3| < \delta$ 'Cause this is the meaning of δ

$|x - 3| < \frac{\varepsilon}{7}$ Substitution

$|x - 3||x + 3| < \frac{\varepsilon}{7}|x + 3|$ Multiply by $|x + 3|$

$|x^2 - 9| < \frac{\varepsilon}{7} * 7$ Because, again, $5 < |x + 3| < 7$

$|x^2 - 9| < \varepsilon$

$|x^2 - 4 - 5| < \varepsilon$

$|f(x) - 5| < \varepsilon$ Again, this is where we want to be.

Done!

We have now shown that $|f(x) - 5| < \varepsilon$ for each of our two cases: $|x - 3| < 1$ and $|x - 3| < \frac{\varepsilon}{7}$; therefore, $\lim_{x\to 3}(x^2 - 4) = 5$.