Directional Derivatives and Gradients

Directional Derivatives, $D_{\mathbf{u}}$

f(x, y)

Given $f(x,y)$ and unit vector $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$, the directional derivative of f in the direction of $\mathbf{u}$ is:

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

f(x, y, z)

Given $f(x,y,z)$ and unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, the directional derivative of f in the direction of $\mathbf{u}$ is:

$$D_{\mathbf{u}}f(x,y) = a f_x(x,y,z) + b f_y(x,y,z) + c f_z(x,y,z)$$

Gradient, ∇

$$\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}$$

$$\nabla f(x,y,z) = f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}$$

Gradient Properties

The following are true for functions of two or three independent variables:

- If $\nabla f(x,y) = \mathbf{0}$, then $D_{\mathbf{u}}f(x,y) = 0$ for all $\mathbf{u}$.
- The direction of maximum increase of $f(x,y)$ is $\nabla f(x,y)$.
- The direction of minimum increase of $f(x,y)$ is $-\nabla f(x,y)$.
  - ▹ Think of this as the direction of maximum decrease.
- The value of the maximum increase is $\|\nabla f(x,y)\|$.
- The value of the minimum increase is $-\|\nabla f(x,y)\|$.
- $\nabla f(x_0,y_0)$ is normal to the level curve passing through (x_0, y_0).