Tests for Convergence

The following are presented in a reasonable order for testing a series for convergence.

Geometric Series

Given a geometric series of the form $\sum_{n=0}^{\infty} ar^n$, the series converges if $|r| < 1$

Nth-Term Test

Given $\sum_{n=1}^{\infty} a_n$ and $\lim_{n\to\infty} a_n = L$

- If $L = 0$, the series converges
- If $L \neq 0$, the series diverges

Telescoping Series

Given telescoping series $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \ldots + (b_n - b_{n+1})$

- Series converges if $\lim_{n\to\infty} b_{n+1}$ is finite
- Further, the sum of the series, $S = b_1 - \lim_{n\to\infty} b_{n+1}$

p-Series

Given a p-series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$

The series converges if $p > 1$.

Alternating Series

Given alternating series $\sum_{n=0}^{\infty} (-1)^n a_n$ and $\sum_{n=0}^{\infty} (-1)^{n+1} a_n$

where $a_n > 0$

The series converge when two conditions are met:

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \leq a_n$ for all n.

Absolute convergence

If $\sum |a_n|$ converges, then $\sum a_n$ also converges. (Note the converse is not true.)

Absolute vs Conditional

Given convergent series Σa_n

- Σa_n is *absolutely convergent* if $\Sigma|a_n|$ is also convergent.
- Σa_n is *conditionally convergent* if $\Sigma|a_n|$ is divergent.

Integral Test

Given $a_n = f(n)$, where $f(x)$ is positive, continuous, and decreasing for all n,

$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} a_n$ are either both convergent or both divergent.

Root Test

Given $\sum a_n$ and $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

- If $L < 1$, the series converges
- If $L > 1$, the series diverges
- If $L = 1$, the test is inconclusive

Ratio Test

Given $\sum a_n$ (where a_n is non-zero for all n) and $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

- If $L < 1$, the series converges
- If $L > 1$, the series diverges
- If $L = 1$, the test is inconclusive

Direct Comparison Test

Given $0 < a_n \leq b_n$ for all values of n,

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges

Limit Comparison Test

Given $a_n, b_n > 0$ and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$

If L is finite and positive, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge