Advanced Factoring

Remainder Theorem

If you divide a polynomial by $x - k$, the remainder will be $f(k)$.

Rational Zero Test

The rational zeros of a polynomial will always be $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. If none of the values of $\frac{p}{q}$ evaluate to zero, then the function has no rational zeros.

Example

Given $f(x) = 3x^3 - 8x^2 - 25x - 4$, what are its possible rational zeros?

The factors of the constant term, 4, are ±1, ±2, ±4

The factors of the leading coefficient, 3, are ±1, ±3

Thus, the possible zeros are $\pm\frac{1}{1}, \pm\frac{1}{3}, \pm\frac{2}{1}, \pm\frac{2}{3}, \pm\frac{4}{1}, \pm\frac{4}{3}$

Descarte's Rule of Signs

Given a polynomial $f(x)$ whose constant value (the one without an x) is not zero:

- The *number of positive real zeros* is equal to either the number of times the signs of the coefficients of $f(x)$ change as you go from left to right through the polynomial or is less than that number by an even amount.
- The *number of negative real zeros* is equal to either the number of times the signs of the coefficients of $f(-x)$ change as you go from left to right through the function or less than that number by an even amount.
- The *number of complex roots* is equal to the total number of roots (equal to the degree of the function) minus the total real roots.

Upper and Lower Bound Rules

Given a polynomial $f(x)$ with a positive leading coefficient. If you divide $f(x)$ by $(x - c)$ using synthetic division:

- If c is positive and each number in the result of the division is either positive or zero, then c is an upper bound for the real zeros of f. That is, the real zeros will be less than or equal to c.
- If c is negative and the numbers in the result of the division are alternating negative and positive (with zeros counting as either), then c is a lower bound for the real zeros of f.

Finding the Roots of a Polynomial, Step-by-Step

Given a 3rd-order or higher polynomial, here's the process of finding its roots:

1. (Optional) *Use Descarte's Rule of Signs* to find the possible number of positive and negative roots.
2. *Use the Rational Roots Test* to derive a collection of possible roots in the form $\frac{p}{q}$.
3. *Plug the possible roots from step 2 into the original equation* until you find an actual root. (*Hint:* start with +1 and –1.)
   a. If there are only a relatively few possible roots, just continue this until you've accumulated all the roots predicted by Descarte. Otherwise, proceed to step 4.
   b. Speaking of which, don't lose sight of the numbers of positive and negative roots from step 1. If Descarte predicts there will be no negative roots, don't waste time testing the negative values from step 2.
   c. And, finally, pay attention to the upper and lower bound rules. If one of the roots turns out to be a bound, you can exclude from your search all of the possible roots beyond the bounds.
4. *Divide the root from step 3 into the original equation* (using synthetic division) to create a simpler equation.
   a. If the new equation is a quadratic, then solve it by factoring, completing the square, or using the quadratic formula.
   b. If the new equation is not a quadratic, go back to step 3, testing the values that haven't yet been shown to be root.